Chern-Simons Theory, Topological String, and Knot Contact Homology

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- Knot Contact Homology
  - Legendrian DGAs
  - Link invariants

- Physics Results
  - Chern-Simons theory and topological string
  - The conifold and large N transition

- Let $K \subset S^3$ be a knot. The Lagrangian conormal of K is

$$L_K = \{p \in T_K^* S^3 : p|_{TK} = 0\} \approx S^1 \times \mathbb{R}^2.$$

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$$L_K = \{p \in T_K^* S^3 \colon p|_{TK} = 0\} \approx S^1 \times \mathbb{R}^2.$$

- The Legendrian conormal is the ideal boundary of L_K:

$$\Lambda_K = L_K \cap U^* S^3,$$

where $U^* S^3$ is the unit conormal

$$\{(q, p) \in T^* S^3 \colon |p| = 1\} \approx S^3 \times S^2,$$

with the contact form $\alpha = pdq$,

$$\Lambda_K \approx S_x^1 \times S_p^1 \approx T^2,$$

where S_x^1 is the longitude and S_p^1 the meridian.

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- The Reeb vector field R of α: $d\alpha(R, \cdot) = 0$ and $\alpha(R) = 1$. For $\alpha = p\, dq$, the flow of R is the geodesic flow. A flow line of R beginning and ending on a Legendrian submanifold is a Reeb chord of Λ. For Λ_K Reeb chords correspond to binormal geodesic chords on K, and are the critical points of the action functional $\gamma \mapsto \int_\gamma \alpha$ for curves γ with endpoints on Λ_K.

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- The grading $|c|$ of a Reeb chord c is defined by a Maslov index. For binormal geodesics in knot contact homology the grading is the Morse index: min = 0, sad = 1, max = 2.

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- The symplectization of U^*S^3 is $U^*S^3 \times \mathbb{R}$ with symplectic form $d(e^t\alpha)$ where $t \in \mathbb{R}$. Fix an almost complex structure J such that $J(\partial_t) = R$, $J(\ker \alpha) = \ker \alpha$, and $d\alpha(v, Jv) > 0$.

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- If c is a Reeb chord of Λ_K then $c \times \mathbb{R}$ is a J-holomorphic strip with boundary on the Lagrangian submanifold $\Lambda_K \times \mathbb{R}$.

- The Legendrian contact homology algbera of Λ_K is the free unital (non-commutative) algebra

$$\mathcal{A}(\Lambda_K) = \mathbb{C}[H_2(U^*S^3, \Lambda_K)]\langle \text{ Reeb chords } \rangle$$
$$= \mathbb{C}[e^{\pm x}, e^{\pm p}, Q^{\pm 1}]\langle \text{ Reeb chords } \rangle$$

(This definition really applies to the ambient space $\mathbb{R}^3$ rather than S^3, we will see later that the contact homology in low degrees do not see the difference.)
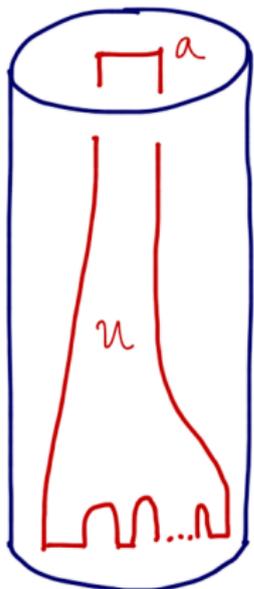
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- $\mathcal{A}(\Lambda_K)$ is a DGA. The differential $\partial\colon \mathcal{A}(\Lambda_K) \to \mathcal{A}(\Lambda_K)$ is linear, satisfies Leibniz rule, decreases grading by 1, and is defined on generators through a holomorphic curve count. The DGA $(\mathcal{A}(\Lambda_K), \partial)$ is invariant under deformations up to homotopy and in particular up to quasi-isomorphism.

$$u : (D, \partial D) \longrightarrow (\mathbb{R} \times Y, \mathbb{R} \times \Lambda),$$

$$du + J \cdot du \cdot i = 0.$$

$$\partial a = \sum_{|a| - |\underline{b}| = 1} |\mathcal{M}_A(a; \underline{b})| \, e^A \, \underline{b}$$

$$b_1 \quad b_2 \cdots b_\kappa \; ; \; \underline{b} = b_1 \ldots b_\kappa$$

- For an alternative definition of the Legendrian DGA with $Q = e^p = 1$ ($Q = e^x = 1$), we consider the wrapped Floer cohomology complex of L_K ($M_K \approx S^3 - K$):

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- Counting solutions to the corresponding parameterized problem gives a DGA which is quasi-isomorphic to the Legendrian DGA defined above.

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- In the limit as $K \to U$, holomorphic disks with boundary on Λ_K admits a description in terms of holomorphic disks with boundary on Λ_U with flow trees determined by $\Lambda_K \subset J^1(\Lambda)$ attached along their boundaries.

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- The unknot
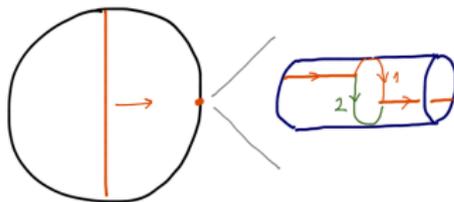
$$\mathcal{A}(\Lambda_U) = \mathbb{C}[e^{\pm x}, e^{\pm p}, Q^{\pm 1}]\langle c, e \rangle, \quad |c| = 1, \ |e| = 2,$$
$$\partial e = c - c = 0, \quad \partial c = 1 - e^x - e^p + Qe^x e^p$$

- The right handed trefoil (differential in degree 1):

$$\mathcal{A}(\Lambda_T) = \mathbb{C}[e^{\pm x}, e^{\pm p}, Q^{\pm 1}]\langle a_{12}, a_{21}, b_{12}, b_{21}, c_{ij}, e_{ij}\rangle_{i,j\in\{1,2\}},$$
$$|a_{ij}| = 0, \ |b_{ij}| = |c_{ij}| = 1, \ |e_{ij}| = 2,$$

$$\partial b_{12} = e^{-x}a_{12} - a_{21},$$
$$\partial b_{21} = e^{x}a_{21} - a_{12},$$
$$\partial c_{11} = e^{p}e^{x} - e^{x} - (2Q - e^{p})a_{12} - Qa_{12}^{2}a_{21},$$
$$\partial c_{12} = Q - e^{p} + e^{p}a_{12} + Qa_{12}a_{21},$$
$$\partial c_{21} = Q - e^{p} + e^{p}e^{x}a_{21} + Qa_{12}a_{21},$$
$$\partial c_{22} = e^{p} - 1 - Qa_{21} + e^{p}a_{12}a_{21},$$

A twist in the braid between the kth and $(k+1)^{\mathrm{th}}$ affects the stable manifolds of the local braid in a standard way. That give rise to certain twist homomorphism of the algebra of degree 0 chords a_{ij} that are linear changes of coordinates except for two quadratic terms. Which give rise to explicit matrices Φ_B^L and Φ_B^R that gives the effect of these isomorphisms on an algebra corresponding to a braid with one additional trivial strand. Assembling a_{ij}, b_{ij}, c_{ij}, and e_{ij} into matrices one then has:

Theorem

The differential on $A(\Lambda_K)$ is determined by the following matrix equations

$$\partial \mathbf{A} = 0,$$
$$\partial \mathbf{B} = -\boldsymbol{\lambda}^{-1} \cdot \mathbf{A} \cdot \boldsymbol{\lambda} + \Phi_B^L \cdot \mathbf{A} \cdot \Phi_B^R,$$
$$\partial \mathbf{C} = \mathbf{A} \cdot \boldsymbol{\lambda} + \mathbf{A} \cdot \Phi_B^R,$$
$$\partial \mathbf{E} = \mathbf{B} \cdot (\Phi_B^R)^{-1} + \mathbf{B} \cdot \boldsymbol{\lambda}^{-1} - \Phi_B^L \cdot \mathbf{C} \cdot \boldsymbol{\lambda}^{-1} + \boldsymbol{\lambda}^{-1} \cdot \mathbf{C} \cdot (\Phi_B^R)^{-1}.$$

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- Consider the standard contact structure on $\mathbb{R}^3$ given by the 1-form α. Then $\mathbb{R} \times \Gamma_\alpha \subset \mathbb{R} \times U^*\mathbb{R}^3$ is a J-holomorphic subvariety for suitable J that is disjoint from the conormal lift of a transverse knot. This induces a filtration on the knot contact homology which gives a non-trivial invariant of transverse knots.

- M closed 3-manifold, A $SU(N)$-connection on M;
  Chern-Simons action:

$$S(A) = \int_M \text{tr} \left(A \wedge dA + \tfrac{2}{3} A \wedge A \wedge A \right)$$

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- From $c_2 = \frac{1}{8\pi^2} \int_X \text{tr} \, F_A^2$, for closed 4-manifolds X, $F_A = dA + A \wedge A$: if $A' = g^{-1}Ag + g^{-1}(dg)$ then $S(A') = S(A) + 8\pi^2 m$, m integer

- For $A \mapsto A + \delta A$, $\delta S = \int_M \text{tr}(F_A \wedge \delta A)$ and flat connections are stationary for S.

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- The Chern-Simons partition function is given by the path integral (over gauge orbits)

$$Z_{CS}(M; N, k) = \int \mathcal{D}A \, e^{\frac{ik}{4\pi} S(A)}, \quad k \text{ integer } (\approx \tfrac{1}{\hbar}).$$

Chern-Simons theory

- Feynman calculus gives a perturbation expansion of Z_{CS} near the flat connections in $\frac{1}{k}$. Interaction term $A^i_j \wedge A^j_k \wedge A^k_i$ gives repeated use of the same vertex and by fattening the diagram we can keep track of the contribution: diagrams with h boundary components and r loops contributes $N^h k^{1-r} = N^h \frac{1}{k}^{-\chi}$.
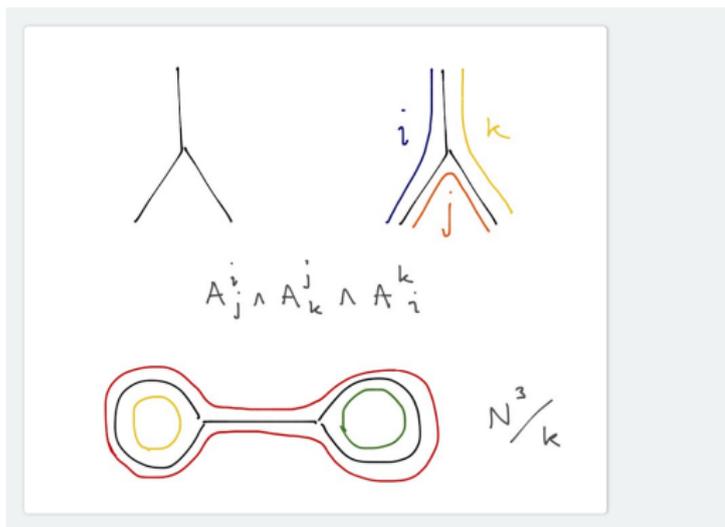
- Witten showed that the Jones and HOMFLY polynomials of knot theory can be interpreted in Chern-Simons theory.

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- $K \subset S^3$ knot, $U(K)$ monodromy of A along K, then with

$$\langle U(K) \rangle = Z_{CS}(S^3; N, k)^{-1} \int \mathcal{D}A \, e^{\frac{ik}{4\pi} S(A)} \operatorname{tr} U(K),$$

$\langle U(K) \rangle = \overline{P}(K)\big|_{v=q^{N/2}, \, z=q^{\frac{1}{2}}-q^{-\frac{1}{2}}}$, where $q = \exp\left(\frac{2\pi i}{N+k}\right)$ and

$$\overline{P}(\text{unknot}) = (q^{N/2} - q^{-N/2})/(q^{1/2} - q^{-1/2}).$$

A-model topological string on T^*M

- The A-model topological string on T^*M with N branes along $M \subset T^*M$ is defined by a path integral over $\Phi = (\phi, \chi, \psi)$, where
    - ϕ is a map of a Riemann surface $\phi \colon (\Sigma, \partial\Sigma) \to (T^*M, M)$,
    - χ a vector field along ϕ (super partner of ϕ),
    - ψ a section in $T^{0,1}(T^*M) \otimes T^{*1,0}\Sigma \oplus T^{1,0}(T^*M) \otimes T^{*0,1}\Sigma$ (super partner of $d\phi$).

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- The Lagrangian is

$$L = 2t \int_\Sigma d^2z \Big(\tfrac{1}{2} G_{ij} \partial_z \phi^i \partial_{\bar z} \phi^j + i G_{I\bar J} \psi_{\bar z}^I D_z \chi^{\bar J} + i G_{\bar I J} \psi_z^{\bar I} D_{\bar z} \chi^I \\ - R_{I\bar I J\bar J} \psi_{\bar z}^I \psi_z^{\bar I} \chi^J \chi^{\bar J} \Big).$$

Here G is the metric $\omega(J\cdot, \cdot)$ and the covariant derivative D_α is

$$D_\alpha \chi^i = \partial_\alpha \chi^i + \partial_\alpha \phi^j \Gamma_{jk}^i \chi^k,$$

- The path integral

$$Z = \int \mathcal{D}\Phi e^L$$

  localizes on holomorphic curves. Averaging over all metrics we get a closed form on $\mathcal{M}_{g,h}$ of the surface. Integrating, we get the desired free energy $F_{g,h}$ for surfaces of genus g with h boundary components.

- In the case of a Calabi-Yau 3-fold the path integral then counts holomorphic maps and its partition function is:

$$Z_{GW}(N, g_s) = \exp\left(\sum_{\substack{g-\text{genus} \\ h-\text{holes}}} F_{g,h} N^h g_s^{2g-2+h} \right),$$

  (where g_s is the string coupling constant $g_s = e^{\phi_0}$, ϕ_0 energy of dilaton field.)

- There are only constant holomorphic maps into (T^*M, M), and the 3-manifold M of constant maps is degenerate. Witten, using string field theory, showed that

$$Z_{CS}(N, k) = Z_{GW}(N, g_s = \frac{2\pi i}{k+N}),$$

as indicated by the fattened Feynman diagrams.

- Take $M = S^3$ in what follows. Let $K \subset S^3$ be a knot. In the A-model on $X = T^*S^3$, add one brane along L_K, get new partition function:

$$Z_{GW}(X, L_K; N, g_s, x),$$

where e^x is the $U(1)$-monodromy around the generator of $H_1(L_K)$.

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where e^x is the $U(1)$-monodromy around the generator of $H_1(L_K)$.

- In Chern-Simons theory, integrating out the strings in $L_K \cap S^3 = K$ corresponds to inserting $\det(1 - e^{-x} U(K))^{-1}$, $U(K)$ the monodromy along K, into the path integral.

- Expanding

$$\det(1 - e^{-x}U(K))^{-1} = \sum_k \text{tr}_{S_k} U(K) \, e^{-kx}$$
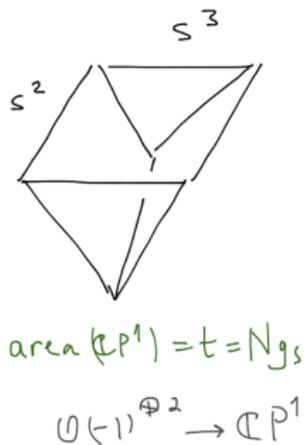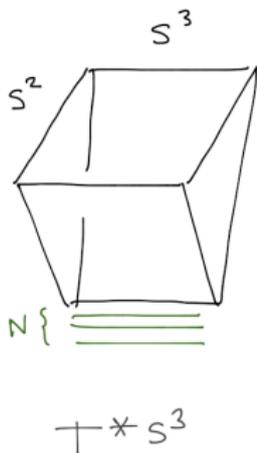
gives

$$\begin{aligned}
\Psi_K(x) &:= Z_{GW}(X, L_K)/Z_{GW}(X) \\
&= \sum_k \langle \text{tr}_{S_k} U(K) \rangle e^{-kx} = \sum_k H_k(K) \, e^{-kx},
\end{aligned}$$

where $H_k(K)$ is a HOMFLY-invariant which is polynomial in $q = e^{g_s}$ and $Q = q^N$.

- $X = T^*S^3$ is a quadric in $\mathbb{C}^4$, a resolution of a cone. There is another resolution Y, total space of $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{C}P^1$.

S^3

S^2

S^3

S^2

$N \{$

area $(\mathbb{C}P^1) = t = N g_s$

$T^* S^3$

$\mathcal{O}(-1)^{\oplus 2} \to \mathbb{C}P^1$

- Gopakumar-Vafa proposes: if area($\mathbb{C}P^1$) $= t = Ng_s$, and $Q = e^{-t} = q^N$, then

$$Z_{GW}(X; N, g_s) = Z_{GW}(Y; g_s, Q),$$

relating A-model open strings in X to A-model closed strings in Y.

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- L_K can be shifted off the 0-section by a non-exact Lagrangian isotopy. Thus, $L_K \subset Y$, and analogously

$$\Psi_K(x) = Z_{GW}(Y; L_K)/Z_{GW}(Y)$$

- Witten's argument relates constant curves on L_K to $GL(1)$-gauge theory on the solid torus which corresponds to ordinary QM in the periods of the connection. This gives

$$p\Psi_K(x) = g_s \frac{\partial}{\partial x}\Psi_K(x),$$

and the usual asymptotics:

$$\Psi_K(x) = \exp\left(\frac{1}{g_s}\int p\,dx + \dots\right).$$

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- From the GW-perspective

$$\Psi_K(x) = \exp\left(\frac{1}{g_s}W_K(x) + \dots\right),$$

where $W_K(x)$ is the disk potential of L_K and where $\dots$ counts curves with $2g - 2 + h > -1$.

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- This equation gives a local parameterization of an algebraic curve that in all examples agree with the augmentation variety (and which is important from a mirror symmetry perspective).

- Consider $\mathcal{A} = \mathcal{A}(\Lambda_K)$ as a family over $(\mathbb{C}^*)^3$ of $\mathbb{C}$-algebras, where points in $(\mathbb{C}^*)^3$ correspond to values of coefficients (e^x, e^p, Q).

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- An augmentation of $\mathcal{A}$ is a chain map $\epsilon \colon \mathcal{A} \to \mathbb{C}$, $\epsilon \circ \partial = 1$, of DGAs, where we consider $\mathbb{C}$ as a DGA with trivial differential concentrated in degree 0.

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- The augmentation variety V_K is the algebraic closure of

$$\left\{ (e^x, e^p, Q) \in (\mathbb{C}^*)^3 \colon \mathcal{A} \text{ has augmentation} \right\}.$$

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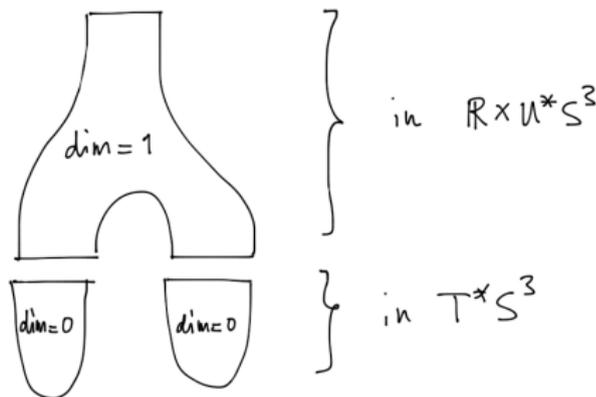
- The trefoil T:

$$\begin{aligned}
A_T(e^x, e^p, Q) = &(e^{4p} - e^{3p})e^{2x} \\
&+ (e^{4p} - Qe^{3p} + 2Q^2 e^{2p} - 2Qe^{2p} - Q^2 e^p + Q^2)e^x \\
&+ (-Q^3 e^p + Q^4).
\end{aligned}$$

- Exact Lagrangian fillings L of Λ_K in T^*S^3 induces augmentations by

$$\epsilon(a) = \sum_{|a|=0} |\mathcal{M}_A(a)| A.$$

The map on coefficients are just the induced map on homology.

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- For the unknot $A_U(e^x, e^p, Q = 1) = (1 - e^x)(1 - e^p)$.

- For the trefoil $A_T(e^x, e^p, Q = 1) = (1 - e^x)(1 - e^p)(e^{3p} - 1)$.